

# Long-time behavior of stable-like processes

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December 12, 2012

## Abstract

In this paper, we consider a long-time behavior of stable-like processes. A stable-like process is a Feller process given by the symbol  $p(x, \xi) = -i\beta(x)\xi + \gamma(x)|\xi|^{\alpha(x)}$ , where  $\alpha(x) \in (0, 2)$ ,  $\beta(x) \in \mathbb{R}$  and  $\gamma(x) \in (0, \infty)$ . More precisely, we give sufficient conditions for recurrence, transience and ergodicity of stable-like processes in terms of the stability function  $\alpha(x)$ , the drift function  $\beta(x)$  and the scaling function  $\gamma(x)$ . Further, as a special case of these results we give a new proof for the recurrence and transience property of one-dimensional symmetric stable Lévy processes with the index of stability  $\alpha \neq 1$ .

**Keywords and phrases:** ergodicity, Foster-Lyapunov criteria, Harris recurrence, recurrence, stable-like process, transience

## 1 Introduction

The recurrence and transience property of Lévy processes, and in particular of stable Lévy processes, has been completely studied in [Sat99, Chapter 7]. In this paper, we consider long-time behavior of stable-like processes. Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$ ,  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  and  $\gamma : \mathbb{R} \rightarrow (0, \infty)$  be arbitrary bounded and continuously differentiable functions with bounded derivatives, such that  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\} \leq \sup\{\alpha(x) : x \in \mathbb{R}\} < 2$  and  $0 < \inf\{\gamma(x) : x \in \mathbb{R}\}$ . Under this assumptions, R. Bass [Bas88], R.L. Schilling and J. Wang [SW12, Theorem 3.3.] have shown that there exists a unique Feller process called a *stable-like process*, which we denote by  $\{X_t^\alpha\}_{t \geq 0}$ , given by the following infinitesimal generator

$$\mathcal{A}^\alpha f(x) = \beta(x)f'(x) + \int_{\mathbb{R}} (f(y+x) - f(x) - f'(x)y1_{\{|z| \leq 1\}}(y)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy, \quad (1.1)$$

where

$$c(x) = \gamma(x) \frac{\alpha(x)2^{\alpha(x)-1}\Gamma(\frac{\alpha(x)+1}{2})}{\pi^{\frac{1}{2}}\Gamma(1 - \frac{\alpha(x)}{2})}.$$

Clearly, the symbol of this process is given by  $p(x, \xi) = -i\beta(x)\xi + \gamma(x)|\xi|^{\alpha(x)}$ . The aim of this paper is to find sufficient conditions for recurrence, transience and ergodicity of stable-like processes. The main tool used in proving these conditions is the Foster-Lyapunov criteria for general Markov processes, which were developed in [MT93b].

Long-time behavior of stable-like processes has already been considered in literature. Clearly, in the case when  $\alpha(x)$ ,  $\beta(x)$  and  $\gamma(x)$  are constant functions, the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  becomes one-dimensional symmetric stable Lévy process with the drift. By the Chung-Fuchs criterion (see [Sat99, Corollary 37.17]), its recurrence and transience property depends only on the index of stability  $\alpha \in (0, 2]$  and the drift  $\beta \in \mathbb{R}$ . More precisely, one-dimensional symmetric stable Lévy process with the drift is recurrent if and only if either  $1 < \alpha \leq 2$  and  $\beta = 0$  or  $\alpha = 1$ . In this paper, in the case when  $\alpha \neq 1$  and  $\beta = 0$ , we prove the same fact by using a different technique.

In the general case, R.L. Schilling and J. Wang [SW12, Theorem 1.1 (ii)] have developed a Chung-Fuchs type condition for transience for “nice” Feller processes. By applying this condition to the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$ , they have shown that  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$  is a sufficient condition for transience. In this paper, we relax the assumption  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$ , that is, we give a sufficient condition for transience without any further assumptions on the function  $\alpha(x)$ .

Next, J. Wang [Wan08] has given sufficient conditions for recurrence and ergodicity of general one-dimensional Lévy type Feller processes, that is, the Feller processes given by the following infinitesimal generator

$$\mathcal{A}f(x) = b(x)f'(x) + \frac{1}{2}a(x)f''(x) + \int_{\mathbb{R}} (f(y+x) - f(x) - f'(x)y1_{\{|z| \leq 1\}}(y)) \nu(x, dy),$$

where  $a(x) \geq 0$  and  $b(x) \in \mathbb{R}$  are Borel measurable functions and  $\nu(x, \cdot)$  is a  $\sigma$ -finite Borel kernel on  $\mathbb{R} \times \mathcal{B}(\mathbb{R})$ , such that  $\nu(x, \{0\}) = 0$  and  $\int_{\mathbb{R}} (1 \wedge y^2) \nu(x, dy) < \infty$  holds for all  $x \in \mathbb{R}$ . By applying these conditions to the stable-like case, he has shown that

$$\text{if } \alpha(x) > 1 \quad \text{and} \quad \frac{c(x)}{\alpha(x) - 2} + x\beta(x) + \frac{2c(x)|x|^{2-\alpha(x)}}{\alpha(x) - 1} \leq 0$$

for all  $|x|$  large enough, then the corresponding stable-like process is recurrent (see [Wan08, Theorem 1.4 (i)]). Note that the above recurrence condition does not cover the zero drift case and the case when  $\alpha(x) \leq 1$ . In particular, it does not cover a symmetric stable Lévy process case. Further, conditions for ergodicity presented in that paper do not cover the stable-like case (see [Wan08, Theorem 1.4 (ii)]). In this paper, we give a sufficient condition for recurrence without any further assumptions on the function  $\alpha(x)$  and a sufficient condition for ergodicity in the case when  $1 < \inf\{\alpha(x) : x \in \mathbb{R}\}$ .

Furthermore, in the case when  $\alpha(x)$  and  $\gamma(x)$  are periodic functions with the same period and when  $\beta(x) = 0$ , B. Franke [Fra06, Fra07] has shown that the recurrence and transience property of the corresponding stable-like process depends only on the minimum of the function  $\alpha(x)$ , that is, if the set  $\{x \in \mathbb{R} : \alpha(x) = \alpha_0 := \inf_{x \in \mathbb{R}} \alpha(x)\}$  has positive Lebesgue measure, then the corresponding stable-like process is recurrent if and only if  $\alpha_0 \geq 1$ .

Finally, if the functions  $\alpha(x)$ ,  $\beta(x)$  and  $\gamma(x)$  are of the form

$$\alpha(x) = \begin{cases} \alpha, & x < -k \\ \beta, & x > k, \end{cases} \quad \beta(x) = 0 \quad \text{and} \quad \gamma(x) = \begin{cases} \gamma, & x < -k \\ \delta, & x > k, \end{cases}$$

where  $\alpha, \beta \in (0, 2)$ ,  $\gamma, \delta \in (0, \infty)$  and  $k > 0$ , then by using an overshoot approach, B. Böttcher [Böt11] has shown that the corresponding stable-like process is recurrent if and only if  $\alpha + \beta \geq 2$ .

For the Dirichlet forms approach to the problem of recurrence and transience of stable-like processes without the drift term we refer the reader to [Uem02, IU04], and for the discrete-time analogous of stable-like processes and their recurrence and transience property we refer the reader to [San12a, San12b].

Now, let us state the main results of this paper.

**Theorem 1.1.** (i) Let  $\liminf_{|x| \rightarrow \infty} \alpha(x) \geq 1$ . If

$$\limsup_{|x| \rightarrow \infty} \left( \operatorname{sgn}(x) \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} \beta(x) + \pi \operatorname{ctg} \left( \frac{\pi \alpha(x)}{2} \right) \right) < 0, \quad (1.2)$$

then the corresponding stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  is recurrent.

(ii) If

$$\liminf_{|x| \rightarrow \infty} \left( \operatorname{sgn}(x) \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} \beta(x) + \pi \operatorname{ctg} \left( \frac{\pi \alpha(x)}{2} \right) \right) > 0, \quad (1.3)$$

then the corresponding stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  is transient.

The following constant will appear in the statement of the following theorem. For  $\alpha \in (0, 2)$  and  $\theta \in (0, \alpha)$  let

$$E(\alpha, \theta) := \frac{\alpha}{\theta} \sum_{i=1}^{\infty} \binom{\theta}{2i} \frac{2}{2i - \alpha} - \frac{2}{\theta} + \frac{\alpha {}_2F_1(-\theta, \alpha - \theta, 1 + \alpha - \theta; -1) + \alpha {}_2F_1(-\theta, \alpha - \theta, 1 + \alpha - \theta; 1)}{\theta(\alpha - \theta)},$$

where  $\binom{z}{n}$  is the binomial coefficient and  ${}_2F_1(a, b, c; z)$  is the Gauss hypergeometric function (see Section 3 for the definition of this function).

**Theorem 1.2.** Let  $1 < \alpha := \inf\{\alpha(x) : x \in \mathbb{R}\}$ . If

$$\limsup_{\theta \rightarrow \alpha} \limsup_{|x| \rightarrow \infty} \left( \operatorname{sgn}(x) \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} \beta(x) + \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta} + E(\alpha(x), \theta) \right) < 0, \quad (1.4)$$

then the corresponding stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  is ergodic.

Let us give several remarks about Theorems 1.1 and 1.2. Firstly, note that condition (1.4) implies condition (1.2). To see this, note that

$$\pi \operatorname{ctg} \left( \frac{\pi \alpha(x)}{2} \right) < \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta} + E(\alpha(x), \theta)$$

for all  $0 < \theta < \alpha$  and all  $|x|$  large enough. Thus, as it is commented in Section 2, ergodicity implies recurrence. Secondly, let  $0 < \theta < \liminf_{|x| \rightarrow \infty} \alpha(x)$  be arbitrary, then

$$\limsup_{|x| \rightarrow \infty} \left( \operatorname{sgn}(x) \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} \beta(x) + E(\alpha(x), \theta) \right) < 0 \quad (1.5)$$

implies recurrence of the corresponding stable-like process (see the proof of Theorem 1.2). Further, it can be proved that the function  $\theta \mapsto E(\alpha, \theta)$  is strictly increasing, thus we choose  $\theta$  close to zero. Next, from (3.2), (3.3) and (3.4), it is easy to see that

$$\lim_{\theta \rightarrow 0} E(\alpha, \theta) = \pi \operatorname{ctg} \left( \frac{\pi \alpha}{2} \right).$$

Hence, (1.5) becomes (1.2). Thirdly, note that if  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$ , then the corresponding stable like process cannot be recurrent. Fourthly, the assumption  $\liminf_{|x| \rightarrow \infty} \alpha(x) \geq 1$  in Theorem 1.1 (i) is not restrictive, that is, we can state Theorem 1.1 (i) without any assumptions about the function  $\alpha(x)$  (see the proof of Theorem 1.1 (i)). But, if we allow that  $\liminf_{|x| \rightarrow \infty} \alpha(x) < 1$ , then, clearly, (1.2) does not hold. Finally, as a simple consequence of Theorem 1.1 we get a new proof for the well-known recurrence and transience property of Lévy processes.

**Corollary 1.3.** *A one-dimensional stable Lévy process given by the symbol (characteristic exponent)  $p(\xi) = \gamma|\xi|^\alpha$ , where  $\alpha \neq 1$  and  $\gamma \in (0, \infty)$ , is recurrent if and only if  $\alpha > 1$ .*

Note that Theorem 1.1 (i) does not imply the recurrence property of one-dimensional symmetric 1-stable Lévy process since, in this case, the left-hand side in (1.2) equals to zero.

Now, we explain our strategy of proving the main results. The proofs of Theorems 1.1 and 1.2 are based on the *Foster-Lyapunov criteria* (see Theorem 2.3). These criteria are based on finding an appropriate test function  $V(x)$  (positive and unbounded in the recurrent case, positive and bounded in the transient case and positive and finite in the ergodic case), such that  $\mathcal{A}^\alpha V(x)$  is well defined, and a compact set  $C \subseteq \mathbb{R}$ , such that  $\mathcal{A}^\alpha V(x) \leq 0$  in the recurrent case,  $\mathcal{A}^\alpha V(x) \geq 0$  in the transient case and  $\mathcal{A}^\alpha V(x) \leq -1$  in the ergodic case for all  $x \in C^c$ . The idea is to find test function  $V(x)$  such that the associated level sets  $C_V(r) := \{y : V(y) \leq r\}$  are compact sets and such that  $C_V(r) \uparrow \mathbb{R}$ , when  $r \nearrow \infty$ , in the cases of recurrence and ergodicity and  $C_V(r) \uparrow \mathbb{R}$ , when  $r \nearrow 1$ , in the case of transience. In the recurrent case, for the test function we take  $V(x) = \ln(1 + |x|)$ . In the transient case we take  $V(x) = 1 - (1 + |x|)^{-\theta}$ , where  $0 < \theta < 1$  is arbitrary, and in the ergodic case we take  $V(x) = |x|^\theta$ , where  $1 < \theta < \inf_{x \in \mathbb{R}} \alpha(x)$  is arbitrary. Now, by proving that

$$\limsup_{|x| \rightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)}}{c(x)} \mathcal{A}^\alpha V(x) < 0$$

in the recurrent case,

$$\liminf_{|x| \rightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)+\theta}}{c(x)} \mathcal{A}^\alpha V(x) > 0$$

in the transient case and

$$\limsup_{|x| \rightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)-\theta}}{c(x)} (\mathcal{A}^\alpha V(x) + 1) < 0$$

in the ergodic case, the proofs of Theorems 1.1 and 1.2 are accomplished.

Let us remark that a similar approach, using similar test functions, can be found in [Lam60], [MAY95] and [San12a] in the discrete-time case and in [ST97] and [Wan08] in the continuous-time case.

The paper is organized as follows. In Section 2 we recall some preliminary and auxiliary results regarding long-time behavior of general Markov processes and we discuss several structural properties of stable-like processes which will be crucial in finding sufficient conditions for recurrence, transience and ergodic properties. Further, we also give and discuss some consequences of the main results. Finally, in Section 3, using the Foster-Lyapunov criteria, we give proofs of Theorems 1.1 and 1.2.

Throughout the paper we use the following notation. We write  $\mathbb{Z}_+$  and  $\mathbb{Z}_-$  for nonnegative and nonpositive integers, respectively. By  $\lambda(\cdot)$  we denote the Lebesgue measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^d)$ . Furthermore,  $\{\Omega, \mathcal{F}, \{\mathbb{P}^x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0}, \{X_t\}_{t \geq 0}\}$ ,  $\{X_t\}_{t \geq 0}$  in the sequel, will denote an arbitrary Markov process on  $\mathbb{R}^d$  with transition kernel  $p^t(x, \cdot) := \mathbb{P}^x(X_t \in \cdot)$  and  $\{X_t^\alpha\}_{t \geq 0}$  will denote the stable-like process given by the infinitesimal generator (1.1).

## 2 Preliminary and auxiliary results

In this section we recall some preliminary and auxiliary results regarding long-time behavior of general Markov processes and we discuss several structural properties of stable-like processes.

**Definition 2.1.** Let  $\{X_t\}_{t \geq 0}$  be a càdlàg strong Markov process on  $\mathbb{R}^d$ . The process  $\{X_t\}_{t \geq 0}$  is called

- (i) Lebesgue irreducible if  $\lambda(B) > 0$  implies  $\int_0^\infty p^t(x, B)dt > 0$  for all  $x \in \mathbb{R}^d$ .
- (ii) recurrent if it is Lebesgue irreducible and if  $\lambda(B) > 0$  implies  $\int_0^\infty p^t(x, B)dt = \infty$  for all  $x \in \mathbb{R}^d$ .
- (iii) Harris recurrent if it is Lebesgue irreducible and if  $\lambda(B) > 0$  implies  $\mathbb{P}^x(\tau_B < \infty) = 1$  for all  $x \in \mathbb{R}^d$ , where  $\tau_B := \inf\{t \geq 0 : X_t \in B\}$ .
- (iv) transient if it is Lebesgue irreducible and if there exists a countable covering of  $\mathbb{R}^d$  with sets  $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$ , such that for each  $j \in \mathbb{N}$  there is a finite constant  $M_j \geq 0$  such that  $\int_0^\infty p^t(x, B_j)dt \leq M_j$  holds for all  $x \in \mathbb{R}^d$ .

Note that the Lebesgue irreducibility of stable Lévy processes is trivially satisfied, and the Lebesgue irreducibility of general stable-like processes has been shown in [SW12, Theorem 3.3]. Hence, according to [Twe94, Theorem 2.3], every stable-like process is either recurrent or transient. Further, every Harris recurrent process is recurrent but in general, these two properties are not equivalent. They differ on the set of the irreducibility measure zero (see [Twe94, Theorem 2.5]). In the case of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$ , by [Böt11, Theorems 4.2 and 4.4], these two properties are equivalent.

A  $\sigma$ -finite measure  $\pi(\cdot)$  on  $\mathcal{B}(\mathbb{R}^d)$  is called an *invariant measure* for the Markov process  $\{X_t\}_{t \geq 0}$  if

$$\pi(B) = \int_{\mathbb{R}^d} p^t(x, B)\pi(dx)$$

holds for all  $t > 0$  and all  $B \in \mathcal{B}(\mathbb{R}^d)$ . It is shown in [Twe94, Theorem 2.6] that if  $\{X_t\}_{t \geq 0}$  is a recurrent process then there exists a unique (up to constant multiples) invariant measure  $\pi(\cdot)$ . If the invariant measure is finite, then it may be normalized to a probability measure. If  $\{X_t\}_{t \geq 0}$  is (Harris) recurrent with finite invariant measure  $\pi(\cdot)$ , then  $\{X_t\}_{t \geq 0}$  is called *positive (Harris) recurrent*, otherwise it is called *null (Harris) recurrent*. The Markov process  $\{X_t\}_{t \geq 0}$  is called *ergodic* if an invariant probability measure  $\pi(\cdot)$  exists and if

$$\lim_{t \rightarrow \infty} \|p^t(x, \cdot) - \pi(\cdot)\| = 0$$

holds for all  $x \in \mathbb{R}^d$ , where  $\|\cdot\|$  denotes the total variation norm on the space of signed measures. One would expect that every positive (Harris) recurrent process is ergodic, but in general this is not true (see [MT93a]). In the case of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$ , these two properties coincide. Indeed, according to [MT93a, Theorem 6.1] and [SW12, Theorem 3.3] it suffices to show that if  $\{X_t^\alpha\}_{t \geq 0}$  possess an invariant probability measure  $\pi(\cdot)$ , then it is recurrent. Assume that  $\{X_t^\alpha\}_{t \geq 0}$  is transient. Then there exist a countable covering of  $\mathbb{R}$  with sets  $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R})$ , such that for each  $j \in \mathbb{N}$  there is a finite constant  $M_j \geq 0$  such that  $\int_0^\infty p^t(x, B_j)dt \leq M_j$  holds for all  $x \in \mathbb{R}$ . Let  $t > 0$  be arbitrary. Then for each  $j \in \mathbb{N}$  we have

$$t\pi(B_j) = \int_0^t \int_{\mathbb{R}} p^s(x, B_j)\pi(dx)ds \leq M_j.$$

By letting  $t \rightarrow \infty$  we get that  $\pi(B_j) = 0$  for all  $j \in \mathbb{N}$ , which is impossible. Let us remark that a stable Lévy process is never ergodic since the Lebesgue measure satisfies the invariance property.

Due to the above discussion and from Theorems 1.1 (i), 1.2 and [MT93a, Theorem 3.2], we get the following two additional long-time properties of stable-like processes.

**Corollary 2.2.** (i) Under assumptions of Theorem 1.1 (i), for each initial position  $x \in \mathbb{R}$  and each covering  $\{O_n\}_{n \in \mathbb{N}}$  of  $\mathbb{R}$  by open bounded sets we have

$$\mathbb{P}^x \left( \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} \left\{ \int_m^{\infty} 1_{\{X_t^\alpha \in O_n\}} dt = 0 \right\} \right) = 0, \quad x \in \mathbb{R}.$$

In other words, for each initial position  $x \in \mathbb{R}$  the event  $\{X_t^\alpha \in C^c \text{ for any compact set } C \subseteq \mathbb{R} \text{ and all } t \in \mathbb{R}_+ \text{ sufficiently large}\}$  has probability 0.

(ii) Under assumptions of Theorem 1.2, for each initial position  $x \in \mathbb{R}$  and each  $\varepsilon > 0$ , there exists a compact set  $C \subseteq \mathbb{R}$  such that

$$\liminf_{t \rightarrow \infty} \mathbb{E}^x \left[ \frac{1}{t} \int_0^t 1_{\{X_s^\alpha \in C\}} ds \right] \geq 1 - \varepsilon.$$

As already mentioned, the proofs of Theorems 1.1 and 1.2 are based on the Foster-Lyapunov criteria. Let us recall several notions regarding Markov processes we are going to need in the sequel. The *extended domain* of a càdlàg Markov process  $\{X_t\}_{t \geq 0}$  on  $\mathbb{R}^d$  is defined by

$$\mathcal{D}(\tilde{\mathcal{A}}) := \left\{ f(x) \text{ is Borel measurable : there is a Borel measurable function } g(x) \text{ such that } f(X_t) - f(X_0) - \int_0^t g(X_s) ds \text{ is a local martingale under } \{\mathbb{P}^x\}_{x \in \mathbb{R}^d} \right\}.$$

Let us remark that in general, the function  $g(x)$  does not have to be unique (see [EK86, Page 24]). For  $f \in \mathcal{D}(\tilde{\mathcal{A}})$  we define

$$\tilde{\mathcal{A}}f = \left\{ g(x) \text{ is Borel measurable : } f(X_t) - f(X_0) - \int_0^t g(X_s) ds \text{ is a local martingale under } \{\mathbb{P}^x\}_{x \in \mathbb{R}^d} \right\}.$$

We call  $\tilde{\mathcal{A}}$  the *extended generator* of  $\{X_t\}_{t \geq 0}$ . A function  $g \in \tilde{\mathcal{A}}f$  is usually abbreviated by  $\tilde{\mathcal{A}}f(x) := g(x)$ . Clearly, if  $\mathcal{A}$  is the infinitesimal generator of the Markov process  $\{X_t\}_{t \geq 0}$  with the domain  $\mathcal{D}(\mathcal{A})$ , then  $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\tilde{\mathcal{A}})$  and for  $f \in \mathcal{D}(\mathcal{A})$  the function  $\mathcal{A}f(x)$  is contained in  $\tilde{\mathcal{A}}f$  (see [EK86, Proposition IV.1.7]). In the case of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$ , obviously we have

$$\left\{ f \in C^2(\mathbb{R}) : \left| \int_{\{|y| \geq 1\}} (f(y+x) - f(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \right| < \infty \text{ for all } x \in \mathbb{R} \right\} \subseteq \mathcal{D}(\tilde{\mathcal{A}}), \quad (2.1)$$

and for the function  $\tilde{\mathcal{A}}f(x)$  we can take exactly the function  $\mathcal{A}^\alpha f(x)$ , where  $\mathcal{A}^\alpha$  is the infinitesimal generator of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  given by (1.1).

Next, let  $\{X_t\}_{t \geq 0}$  be a càdlàg Markov process on  $\mathbb{R}^d$  and let

$$T_c := \inf\{t \geq 0 : X_t \notin \mathbb{R}^d\} \quad \text{and} \quad T_e := \lim_{n \rightarrow \infty} \inf\{t \geq 0 : |X_t| > n\}.$$

The process  $\{X_t\}_{t \geq 0}$  is called *conservative* if  $\mathbb{P}^x(T_c = \infty) = 1$  for all  $x \in \mathbb{R}^d$  and *non-explosive* if  $\mathbb{P}^x(T_e = \infty) = 1$  for all  $x \in \mathbb{R}^d$ . By [Sch98, Theorem 5.2], the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  is always conservative and then, since it has càdlàg paths, it also non-explosive.

A function  $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is called a *norm-like function* (for a càdlàg Markov process  $\{X_t\}_{t \geq 0}$ ) if  $V \in \mathcal{D}(\tilde{\mathcal{A}})$  and the level sets  $\{x : V(x) \leq r\}$  are precompact sets for each level  $r \geq 0$ .

Finally, a set  $C \in \mathcal{B}(\mathbb{R}^d)$  is called  $\nu_a$ -petite set (for a càdlàg Markov process  $\{X_t\}_{t \geq 0}$ ) if there exist a probability measure  $a(\cdot)$  on  $\mathcal{B}(\mathbb{R}_+)$  and a nontrivial measure  $\nu_a(\cdot)$  on  $\mathcal{B}(\mathbb{R}^d)$  such that

$$\int_0^\infty p^t(x, B) a(dt) \geq \nu_a(B)$$

holds for all  $x \in C$  and all  $B \in \mathcal{B}(\mathbb{R}^d)$ .

Now, we state the Foster-Lyapunov criteria (see [MT93b, Theorems 3.2 and 4.2] and [ST94, Theorem 3.3]).

**Theorem 2.3.** *Let  $\{X_t\}_{t \geq 0}$  be a non-explosive Lebesgue irreducible càdlàg Markov process on  $\mathbb{R}^d$ .*

(i) *If every compact set is a petite set and if there exist a compact set  $C \subseteq \mathbb{R}^d$  with  $\lambda(C) > 0$ , a constant  $d > 0$  and a norm-like function  $V(x)$ , such that*

$$\tilde{A}V(x) \leq d1_C(x), \quad x \in \mathbb{R}^d,$$

*then the process  $\{X_t\}_{t \geq 0}$  is Harris recurrent.*

(ii) *If there exist a bounded Borel measurable function  $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$  and closed sets  $C, D \subseteq \mathbb{R}^d$ ,  $D \subseteq C^c$ , such that*

- (a)  $\lambda(C) > 0$ ,  $\lambda(D) > 0$  and  $\sup_{x \in C} V(x) < \inf_{x \in D} V(x)$
- (b)  $\tilde{A}V(x) \geq 0$  for all  $x \in C^c$ ,

*then the process  $\{X_t\}_{t \geq 0}$  is transient.*

(iii) *If every compact set is a petite set and if there exist  $d > 0$ , a compact set  $C \subseteq \mathbb{R}^d$  with  $\lambda(C) > 0$ , a Borel measurable function  $f(x) \geq 1$  and a nonnegative function  $V \in \mathcal{D}(\tilde{A})$ , such that*

- (a)  $V(x)$  is bounded on  $C$
- (b)  $\tilde{A}V(x) \leq -f(x) + d1_C(x)$  for all  $x \in \mathbb{R}^d$ ,

*then the process  $\{X_t\}_{t \geq 0}$  is positive Harris recurrent and*

$$\pi(f) := \int_{\mathbb{R}^d} f(x) \pi(dx) < \infty,$$

*where  $\pi(\cdot)$  is the invariant measure for  $\{X_t\}_{t \geq 0}$ .*

Let us remark that in the case of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$ , according to [Twe94, Theorems 5.1 and 7.1], the first requirements of Theorem 2.3 (i) and (iii) always hold, that is, every compact set is a petite set.

We end this section with the following observation. Assume that  $\{X_t\}_{t \geq 0}$  is an ergodic Markov process with invariant measure  $\pi(\cdot)$ . Then, clearly,

$$\lim_{t \rightarrow \infty} \mathbb{E}^x[f(X_t)] = \pi(f)$$

holds for all  $x \in \mathbb{R}^d$  and all bounded Borel measurable functions  $f(x)$ . In what follows, we extend this convergence to a wider class of functions. For any Borel measurable function  $f(x) \geq 1$  and any signed measure  $\mu(\cdot)$  on  $\mathcal{B}(\mathbb{R}^d)$  we write

$$\|\mu\|_f := \sup_{|g| \leq f} |\mu(g)|,$$



where the supremum is taken over all Borel measurable functions  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying  $|g(x)| \leq f(x)$  for all  $x \in \mathbb{R}^d$ . A Markov process  $\{X_t\}_{t \geq 0}$  is called *f-ergodic* if it is positive Harris recurrent with invariant probability measure  $\pi(\cdot)$ , if  $\pi(f) < \infty$ , and

$$\lim_{t \rightarrow \infty} \|p^t(x, \cdot) - \mu(f)\|_f = 0, \quad x \in \mathbb{R}^d.$$

Note that  $\|\cdot\|_1 = \|\cdot\|$ . Hence, *f-ergodicity* implies ergodicity. Now, by Theorems 1.2, 2.3 (iii) and [MT93b, Theorem 5.3 (ii)], we get the following sufficient condition for *f-ergodicity*.

**Theorem 2.4.** *Let  $1 < \alpha := \inf_{x \in \mathbb{R}} \alpha(x)$  and let  $\theta \in (1, \alpha)$  be arbitrary. If there exist Borel measurable function  $f(x) \geq 1$  and strictly increasing, nonnegative and convex function  $\phi(x)$ , such that  $|x|^\theta = \phi(f(x))$  for all  $|x|$  large enough and*

$$\limsup_{|x| \rightarrow \infty} \left( \operatorname{sgn}(x) \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} \beta(x) + \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta} f(x) + E(\alpha(x), \theta) \right) < 0, \quad (2.2)$$

*then the corresponding stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  is *f-ergodic* (recall that the constant  $E(\alpha, \theta)$  is defined in Theorem 1.2).*

Now, let  $\eta \in (0, \theta]$  be arbitrary. By taking  $f(x) = |x|^\eta$  and  $\phi(x) = |x|^{\frac{\theta}{\eta}}$  for all  $|x|$  large enough, we get the following corollary.

**Corollary 2.5.** *Let  $1 < \alpha := \inf_{x \in \mathbb{R}} \alpha(x)$ . If*

$$\limsup_{\theta \rightarrow \alpha} \limsup_{|x| \rightarrow \infty} \left( \operatorname{sgn}(x) \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} \beta(x) + \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta+\eta} + E(\alpha(x), \theta) \right) < 0$$

*for some  $\eta \in (0, \alpha)$ , then the corresponding stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  is *f-ergodic* for every function  $f(x) \geq 1$  such that  $f(x) \leq |x|^\eta$  for all  $|x|$  large enough.*

### 3 Proof of the main results

In this section we give proofs of Theorems 1.1 and 1.2. Before the proofs, we recall several special functions we need. The Gamma function is defined by the formula

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \operatorname{Re}(z) > 0.$$

It can be analytically continued on  $\mathbb{C} \setminus \mathbb{Z}_-$  and it satisfies the following two well-known properties

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad \Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}. \quad (3.1)$$

The Digamma function is a function defined by  $\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$ , for  $z \in \mathbb{C} \setminus \mathbb{Z}_-$ , and it satisfies the following properties:

(i)

$$\Psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \quad (3.2)$$

where  $\gamma$  is the Euler's number



(ii)

$$\Psi(1+z) = \Psi(z) + \frac{1}{z} \quad (3.3)$$

(iii)

$$\Psi(1-z) = \Psi(z) + \pi \operatorname{ctg}(\pi z). \quad (3.4)$$

The Gauss hypergeometric function is defined by the formula

$${}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (3.5)$$

for  $a, b, c, z \in \mathbb{C}$ ,  $c \notin \mathbb{Z}_-$ , where for  $w \in \mathbb{C}$  and  $n \in \mathbb{Z}_+$ ,  $(w)_n$  is defined by

$$(w)_0 = 1 \quad \text{and} \quad (w)_n = w(w+1) \cdots (w+n-1).$$

The series (3.5) absolutely converges on  $|z| < 1$ , absolutely converges on  $|z| \leq 1$  when  $\operatorname{Re}(c-a-b) > 0$ , conditionally converges on  $|z| \leq 1$ , except for  $z = 1$ , when  $-1 < \operatorname{Re}(c-b-a) \leq 0$  and diverges when  $\operatorname{Re}(c-b-a) \leq -1$ . In the case when  $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ , it can be analytically continued on  $\mathbb{C} \setminus (1, \infty)$  by the formula

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt, \quad (3.6)$$

and for  $a, b \in \mathbb{C}$ ,  $c \in \mathbb{C} \setminus \mathbb{Z}_-$  and  $z \in \mathbb{C} \setminus (0, \infty)$  it satisfies the following relation

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(a, 1-c+a, 1-b+a; \frac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(b, 1-c+b, 1-a+b; \frac{1}{z}\right). \end{aligned} \quad (3.7)$$

For further properties of the Gamma function, the Digamma function and hypergeometric functions see [AS84, Chapters 6 and 15]. We also need the following two lemmas.

**Lemma 3.1.**

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{x}{1+x} \right)^n = 0.$$

*Proof.* First, note that for  $x \geq 0$ , by the binomial expansion of  $(1+x)^n$ , we have

$$(1+x)^n \geq nx^{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Now, the desired result follows by the dominated convergence theorem.  $\square$

**Lemma 3.2.** *Let  $\alpha : \mathbb{R} \rightarrow (0, 1) \cup (1, 2)$  be an arbitrary function. Then for every  $R \geq 0$  we have*

$$\lim_{x \rightarrow \infty} \frac{1}{1-\alpha(x)} \left( 1 - \left( \frac{x}{x+R} \right)^{1-\alpha(x)} \right) = 0.$$

*Proof.* Let  $0 < \varepsilon < 1$  be arbitrary. Since

$$\frac{1}{t} (1 - (1 - \varepsilon)^t) \leq -2 \ln(1 - \varepsilon)$$

holds for all  $t \in (-1, 0) \cup (0, 1)$ , we have

$$0 \leq \limsup_{x \rightarrow \infty} \frac{1}{1 - \alpha(x)} \left( 1 - \left( \frac{x}{x + R} \right)^{1 - \alpha(x)} \right) \leq \limsup_{x \rightarrow \infty} \frac{1 - (1 - \varepsilon)^{1 - \alpha(x)}}{1 - \alpha(x)} \leq -2 \ln(1 - \varepsilon).$$

Now, by letting  $\varepsilon \rightarrow 0$ , we have the claim.  $\square$

**Proof of Theorem 1.1 (i).** The proof is divided in four steps.

**Step 1.** In the first step we explain our strategy of the proof. Let  $\varphi \in C^2(\mathbb{R})$  be an arbitrary nonnegative function such that  $\varphi(x) = |x|$ , for  $|x| > 1$ , and  $\varphi(x) \leq |x|$ , for  $|x| \leq 1$ . Now, let us define the function  $V : \mathbb{R} \rightarrow \mathbb{R}_+$  by the formula

$$V(x) := \ln(1 + \varphi(x)).$$

Clearly,  $V \in C^2(\mathbb{R})$  and the level set  $C_V(r) := \{x : V(x) \leq r\}$  is a compact set for all levels  $r \geq 0$ . Furthermore, it is easy to see that

$$\left| \int_{\{|y| > 1\}} (V(x + y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \right| < \infty$$

holds for all  $x \in \mathbb{R}$ . Hence, by the relation (2.1),  $V \in \mathcal{D}(\tilde{\mathcal{A}})$  and for the function  $\tilde{\mathcal{A}}V(x)$  we can take the function  $\mathcal{A}^\alpha V(x)$ , where  $\mathcal{A}^\alpha$  is the infinitesimal generator of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  given by (1.1). In the sequel we show that there exists  $r_0 > 0$ , large enough, such that  $\tilde{\mathcal{A}}V(x) \leq 0$  for all  $x \in (C_V(r_0))^c$ . Clearly,  $\sup_{x \in C_V(r_0)} |\tilde{\mathcal{A}}V(x)| < \infty$ . Thus, the desired result follows from Theorem 2.3 (i). In order to see this, since  $C_V(r) \uparrow \mathbb{R}$ , when  $r \nearrow \infty$ , it suffices to show that

$$\limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1 + |x|)^{\alpha(x)} \tilde{\mathcal{A}}V(x) < 0. \quad (3.8)$$

**Step 2.** In the second step, we find more appropriate expression for the function  $\tilde{\mathcal{A}}V(x)$ . We have

$$\begin{aligned} \tilde{\mathcal{A}}V(x) &= \mathcal{A}^\alpha V(x) = \beta(x) V'(x) + \int_{\mathbb{R}} (V(x + y) - V(x) - V'(x)y 1_{\{|y| \leq 1\}}(y)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &= \beta(x) V'(x) + \int_{\{|y| \leq 1\}} (V(x + y) - V(x) - V'(x)y) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + \int_{\{|y| > 1\}} (V(x + y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

Let us define

$$\begin{aligned} A(x) &:= \beta(x) V'(x) \\ B(x) &:= \int_{\{|y| \leq 1\}} (V(x + y) - V(x) - V'(x)y) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ C(x) &:= \int_{\{|y| > 1\}} (V(x + y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

Hence, in order to prove (3.8) it suffices to prove

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1 + |x|)^{\alpha(x)} \tilde{\mathcal{A}}V(x) &= \limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1 + |x|)^{\alpha(x)} (A(x) + B(x) + C(x)) \\ &\leq \limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1 + |x|)^{\alpha(x)} (A(x) + C(x)) + \limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1 + |x|)^{\alpha(x)} B(x) < 0. \end{aligned} \quad (3.9)$$

Furthermore, for  $x > 0$  large enough we have

$$\begin{aligned} A(x) &= \frac{\beta(x)}{1+x}, \\ B(x) &= \int_{\{|y| \leq 1\}} \left( \ln(1+x+y) - \ln(1+x) - \frac{y}{1+x} \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &= \int_{-1}^1 \left( \ln \left( 1 + \frac{y}{1+x} \right) - \frac{y}{1+x} \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \end{aligned}$$

and

$$\begin{aligned} C(x) &= \int_{\{y \leq -x-1\}} (\ln(1-x-y) - \ln(1+x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + \int_{\{-x-1 \leq y \leq -x+1\}} (\ln(1+\varphi(x+y)) - \ln(1+x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + \int_{\{-x+1 \leq y < -1\}} (\ln(1+x+y) - \ln(1+x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + \int_{\{y > 1\}} (\ln(1+x+y) - \ln(1+x)) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &= \int_{1+x}^{\infty} \ln \left( \frac{1-x+y}{1+x} \right) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &\quad + \int_{x-1}^{1+x} \ln \left( \frac{1+\varphi(x-y)}{1+x} \right) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &\quad + \int_1^{x-1} \ln \left( 1 - \frac{y}{1+x} \right) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &\quad + \int_1^{\infty} \ln \left( 1 + \frac{y}{1+x} \right) \frac{c(x)}{y^{\alpha(x)+1}} dy. \end{aligned}$$

**Step 3.** In the third step, we compute  $\limsup_{x \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)} B(x)$ . By using the elementary inequality  $t - t^2 \leq \ln(1+t) \leq t$ , we get

$$-\frac{2c(x)}{(2-\alpha(x))(1+x)^2} = -\frac{c(x)}{(1+x)^2} \int_{-1}^1 |y|^{1-\alpha(x)} dy \leq B(x) \leq 0.$$

Hence,

$$\lim_{x \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)} B(x) = 0. \quad (3.10)$$

**Step 4.** In the fourth step, we compute  $\limsup_{x \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)} (A(x) + C(x))$ . In the case when  $\alpha(x) = 1$ , we have

$$\begin{aligned} \frac{1+x}{c(x)} C(x) &= \ln \left( \frac{1}{1+x} \right) + \frac{\ln \left( \frac{1+x}{4} \right)}{x-1} + \frac{x}{x-1} \ln(1+x) \\ &\quad + (1+x) \int_{x-1}^{1+x} \ln \left( \frac{1+\varphi(x-y)}{1+x} \right) \frac{dy}{y^2} - \ln(x-1) + x \ln(x) \\ &\quad - \frac{\ln \left( \frac{4}{1+x} \right)}{x-1} + \frac{\ln(1+x)}{x-1} + \frac{x}{x-1} \ln(1+x) - \frac{x^2}{x-1} \ln(1+x) \\ &\quad + \ln(2+x) + (1+x) \ln \left( 1 + \frac{1}{1+x} \right). \end{aligned}$$

Now, since

$$\frac{2}{x-1} \ln \left( \frac{1}{x-1} \right) \leq (1+x) \int_{x-1}^{1+x} \ln \left( \frac{1+\varphi(x-y)}{1+x} \right) \frac{dy}{y^2} \leq \frac{2}{x-1} \ln \left( \frac{2}{x-1} \right),$$

by elementary computation we get

$$\lim_{x \rightarrow \infty} \frac{1+x}{c(x)} C(x) = 0. \quad (3.11)$$

Further, in the case when  $\alpha(x) \neq 1$ , using integration by parts formula and (3.6), we get

$$\begin{aligned} &\frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)} C(x) \\ &= \ln \left( \frac{2}{1+x} \right) + \frac{{}_2F_1 \left( 1, \alpha(x), 1 + \alpha(x); \frac{x-1}{1+x} \right)}{\alpha(x)} \\ &\quad + \alpha(x) (1+x)^{\alpha(x)} \int_{x-1}^{1+x} \ln \left( \frac{1+\varphi(x-y)}{1+x} \right) \frac{dy}{y^{\alpha(x)+1}} \\ &\quad + (1+x)^{\alpha(x)} \ln \left( 1 - \frac{1}{1+x} \right) - (1+x)^{\alpha(x)} (x-1)^{-\alpha(x)} \ln \left( \frac{2}{1+x} \right) \\ &\quad + \frac{{}_2F_1 \left( 1, 1 - \alpha(x), 2 - \alpha(x); \frac{x-1}{1+x} \right)}{(\alpha(x) - 1)(1+x)^{1-\alpha(x)}(x-1)^{\alpha(x)-1}} - \frac{{}_2F_1 \left( 1, 1 - \alpha(x), 2 - \alpha(x); \frac{1}{1+x} \right)}{(\alpha(x) - 1)(1+x)^{1-\alpha(x)}} \\ &\quad + (1+x)^{\alpha(x)} \ln \left( 1 + \frac{1}{1+x} \right) + \frac{{}_2F_1 \left( 1, \alpha(x), 1 + \alpha(x); -x-1 \right)}{\alpha(x)(1+x)^{-\alpha(x)}}. \end{aligned}$$

Let us put

$$\begin{aligned}
C_1(x) &:= \ln \left( \frac{2}{1+x} \right) - (1+x)^{\alpha(x)} (x-1)^{-\alpha(x)} \ln \left( \frac{2}{1+x} \right) \\
C_2(x) &:= (1+x)^{\alpha(x)} \ln \left( 1 - \frac{1}{1+x} \right) + (1+x)^{\alpha(x)} \ln \left( 1 + \frac{1}{1+x} \right) \\
C_3(x) &:= \alpha(x) (1+x)^{\alpha(x)} \int_{x-1}^{1+x} \ln \left( \frac{1+\varphi(x-y)}{1+x} \right) \frac{dy}{y^{\alpha(x)+1}} \\
C_4(x) &:= \frac{{}_2F_1 \left( 1, \alpha(x), 1+\alpha(x); -x-1 \right)}{\alpha(x)(1+x)^{-\alpha(x)}} - \frac{{}_2F_1 \left( 1, 1-\alpha(x), 2-\alpha(x); \frac{1}{1+x} \right)}{(\alpha(x)-1)(1+x)^{1-\alpha(x)}} \\
C_5(x) &:= \frac{{}_2F_1 \left( 1, 1-\alpha(x), 2-\alpha(x); \frac{x-1}{1+x} \right)}{(\alpha(x)-1)(1+x)^{1-\alpha(x)}(x-1)^{\alpha(x)-1}} + \frac{{}_2F_1 \left( 1, \alpha(x), 1+\alpha(x); \frac{x-1}{1+x} \right)}{\alpha(x)}.
\end{aligned}$$

Thus,

$$\frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)} C(x) = C_1(x) + C_2(x) + C_3(x) + C_4(x) + C_5(x). \quad (3.12)$$

Now, by applying Lemma 3.2, it is easy to see that

$$\lim_{x \rightarrow \infty} C_1(x) = 0, \quad (3.13)$$

and

$$\lim_{x \rightarrow \infty} C_2(x) = 0. \quad (3.14)$$

Further, since  $0 \leq \varphi(y) \leq 1$ , for  $|y| \leq 1$ , we have

$$\ln \left( \frac{1}{1+x} \right) \left( \left( \frac{1+x}{x-1} \right)^{\alpha(x)} - 1 \right) \leq C_3(x) \leq \ln \left( \frac{2}{1+x} \right) \left( \left( \frac{1+x}{x-1} \right)^{\alpha(x)} - 1 \right).$$

Again by Lemma 3.2, it follows that

$$\lim_{x \rightarrow \infty} C_3(x) = 0. \quad (3.15)$$

Further, by applying (3.1) and (3.7) we get

$$\begin{aligned}
C_4(x) &= \frac{{}_2F_1 \left( 1, 1-\alpha(x), 2-\alpha(x); -\frac{1}{1+x} \right)}{(\alpha(x)-1)(1+x)^{1-\alpha(x)}} + \frac{\pi}{\sin(\alpha(x)\pi)} \\
&\quad - \frac{{}_2F_1 \left( 1, 1-\alpha(x), 2-\alpha(x); \frac{1}{1+x} \right)}{(\alpha(x)-1)(1+x)^{1-\alpha(x)}}.
\end{aligned} \quad (3.16)$$

Now, since  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\} \leq \sup\{\alpha(x) : x \in \mathbb{R}\} < 2$ , by (3.5), we have

$$\lim_{x \rightarrow \infty} \left( \frac{{}_2F_1 \left( 1, 1-\alpha(x), 2-\alpha(x); -\frac{1}{x+1} \right)}{(\alpha(x)-1)(x+1)^{1-\alpha(x)}} - \frac{{}_2F_1 \left( 1, 1-\alpha(x), 2-\alpha(x); \frac{1}{x+1} \right)}{(\alpha(x)-1)(x+1)^{1-\alpha(x)}} \right) = 0. \quad (3.17)$$

Next,

$$C_5(x) = \frac{{}_2F_1\left(1, 1 - \alpha(x), 2 - \alpha(x); \frac{x-1}{x+1}\right)}{(\alpha(x) - 1)(x+1)^{1-\alpha(x)}(x-1)^{\alpha(x)-1}} - \frac{{}_2F_1\left(1, 1 - \alpha(x), 2 - \alpha(x); \frac{x-1}{x+1}\right)}{\alpha(x) - 1} \\ + \frac{{}_2F_1\left(1, 1 - \alpha(x), 2 - \alpha(x); \frac{x-1}{x+1}\right)}{\alpha(x) - 1} + \frac{{}_2F_1\left(1, \alpha(x), 1 + \alpha(x); \frac{x-1}{x+1}\right)}{\alpha(x)}. \quad (3.18)$$

Again, since  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\} \leq \sup\{\alpha(x) : x \in \mathbb{R}\} < 2$ , by applying (3.5) and Lemma 3.1, we get

$$\lim_{x \rightarrow \infty} \left( \frac{{}_2F_1\left(1 - \alpha(x), 1, 2 - \alpha(x); \frac{x-1}{x+1}\right)}{(\alpha(x) - 1)(x+1)^{1-\alpha(x)}(x-1)^{\alpha(x)-1}} - \frac{{}_2F_1\left(1 - \alpha(x), 1, 2 - \alpha(x); \frac{x-1}{x+1}\right)}{\alpha(x) - 1} \right) = 0, \quad (3.19)$$

and from (3.5) we get

$$\frac{{}_2F_1\left(1 - \alpha(x), 1, 2 - \alpha(x); \frac{x-1}{x+1}\right)}{\alpha(x) - 1} + \frac{{}_2F_1\left(1, \alpha(x), 1 + \alpha(x); \frac{x-1}{x+1}\right)}{\alpha(x)} \\ = \frac{1}{\alpha(x) - 1} \sum_{n=0}^{\infty} \frac{1 - \alpha(x)}{1 - \alpha(x) + n} \left(\frac{x-1}{x+1}\right)^n + \frac{1}{\alpha(x)} \sum_{n=0}^{\infty} \frac{\alpha(x)}{\alpha(x) + n} \left(\frac{x-1}{x+1}\right)^n \\ = \frac{1}{\alpha(x) - 1} + \frac{1}{\alpha(x)} + \sum_{n=1}^{\infty} \frac{1 - 2\alpha(x)}{(\alpha(x) + n)(1 - \alpha(x) + n)} \left(\frac{x-1}{x+1}\right)^n \\ = \frac{1}{\alpha(x) - 1} + \frac{1}{\alpha(x)} + \sum_{n=1}^{\infty} \frac{1 - 2\alpha(x)}{(\alpha(x) + n)(1 - \alpha(x) + n)} \\ + \sum_{n=1}^{\infty} \frac{1 - 2\alpha(x)}{(\alpha(x) + n)(1 - \alpha(x) + n)} \left( \left(\frac{x-1}{x+1}\right)^n - 1 \right).$$

Clearly, by the dominated convergence theorem we have

$$\lim_{x \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1 - 2\alpha(x)}{(\alpha(x) + n)(1 - \alpha(x) + n)} \left( \left(\frac{x-1}{x+1}\right)^n - 1 \right) = 0, \quad (3.20)$$

and by using the Taylor series expansion of the function  $\text{ctg}(y)$  we get

$$\frac{1}{\alpha(x) - 1} + \frac{1}{\alpha(x)} + \sum_{n=1}^{\infty} \frac{1 - 2\alpha(x)}{(\alpha(x) + n)(1 - \alpha(x) + n)} = \pi \text{ctg} \left( \frac{\pi \alpha(x)}{2} \right). \quad (3.21)$$

Now, by combining (3.9) - (3.21) we get

$$\limsup_{x \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)} (A(x) + C(x)) \\ = \limsup_{x \rightarrow \infty} \left( \frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)-1} \beta(x) + \pi \text{ctg} \left( \frac{\pi \alpha(x)}{2} \right) \right). \quad (3.22)$$

Finally, by combining (3.8), (3.9), (3.22) and assumption (1.2) we get

$$\limsup_{x \rightarrow \infty} \frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)} \tilde{\mathcal{A}}V(x) < 0.$$

The case when  $x < 0$  is treated in the same way. Therefore, we have proved the desired result.  $\square$

**Proof of Theorem 1.1 (ii).** The proof is divided in three steps.

**Step 1.** In the first step we explain our strategy of the proof. Let  $\varphi \in C^2(\mathbb{R})$  be an arbitrary nonnegative function such that  $\varphi(x) = |x|$ , for  $|x| > 1$ , and  $\varphi(x) \leq |x|$ , for  $|x| \leq 1$ . Let  $\theta \in (0, 1)$  be arbitrary and let us define the function  $V : \mathbb{R} \rightarrow \mathbb{R}_+$  by the formula

$$V(x) := 1 - (1 + \varphi(x))^{-\theta}.$$

Clearly,  $V \in C^2(\mathbb{R})$  and the level set  $C_V(r) = \{x : V(x) \leq r\}$  is a compact set for all levels  $0 \leq r < 1$ . Furthermore, since the function  $V(x)$  is bounded

$$\left| \int_{\{|y|>1\}} (V(x+y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \right| < \infty$$

holds for all  $x \in \mathbb{R}$ . Hence, by the relation (2.1),  $V \in \mathcal{D}(\tilde{\mathcal{A}})$  and for the function  $\tilde{\mathcal{A}}V(x)$  we can take the function  $\mathcal{A}^\alpha V(x)$ , where  $\mathcal{A}^\alpha$  is again the infinitesimal generator of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  given by (1.1). In the sequel we show that there exists  $0 < r_0 < 1$ , such that  $\tilde{\mathcal{A}}V(x) \geq 0$  for all  $x \in (C_V(r_0))^c$ . Clearly,  $\sup_{x \in C_V(r_0)} |\tilde{\mathcal{A}}V(x)| < \infty$ . Thus, the desired result follows from Theorem 2.3 (ii). Note that for the sets  $C, D \subseteq \mathbb{R}$ , defined in Theorem 2.3 (ii), we can take  $C := C_V(r_0)$  and  $D$  is an arbitrary closed set satisfying  $D \subseteq C^c$  and  $\lambda(D) > 0$ . Now, from the continuity of the function  $V(x)$ , we have

$$\sup_{x \in C} V(x) < \inf_{x \in D} V(x).$$

In order to prove the existence of such  $r_0$ , since  $C_V(r) \uparrow \mathbb{R}$ , when  $r \nearrow 1$ , it suffices to show that

$$\liminf_{|x| \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} (1 + |x|)^{\alpha(x)+\theta} \tilde{\mathcal{A}}V(x) > 0. \quad (3.23)$$

**Step 2.** In the second step we find more appropriate expression for the function  $\tilde{\mathcal{A}}V(x)$ . We have

$$\begin{aligned} \tilde{\mathcal{A}}V(x) &= \mathcal{A}^\alpha V(x) = \beta(x)V'(x) + \int_{\{|y| \leq 1\}} (V(x+y) - V(x) - V'(x)y) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + \int_{\{|y| > 1\}} (V(x+y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

Let us define

$$\begin{aligned} A(x) &:= \beta(x)V'(x) \\ B(x) &:= \int_{\{|y| \leq 1\}} (V(x+y) - V(x) - V'(x)y) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ C(x) &:= \int_{\{|y| > 1\}} (V(x+y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

Hence, in order to prove (3.23) it suffices to prove

$$\liminf_{|x| \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} (1 + |x|)^{\alpha(x)+\theta} (A(x) + B(x) + C(x)) > 0. \quad (3.24)$$



Furthermore, for  $x > 0$  large enough we have

$$A(x) = \theta \beta(x)(1+x)^{-\theta-1}$$

$$B(x) = \int_{-1}^1 \left( (1+x)^{-\theta} - (1+x+y)^{-\theta} - \theta(1+x)^{-\theta-1}y \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy$$

and

$$\begin{aligned} C(x) &= \int_{\{|y|>1\}} \left( (1+x)^{-\theta} - (1+\varphi(x+y))^{-\theta} \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &= (1+x)^{-\theta} \int_{-\infty}^{-x-1} \left( 1 - \left( \frac{1-x-y}{1+x} \right)^{-\theta} \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + (1+x)^{-\theta} \int_{-x-1}^{-x+1} \left( 1 - \left( \frac{1+\varphi(x+y)}{1+x} \right)^{-\theta} \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + (1+x)^{-\theta} \int_{-x+1}^{-1} \left( 1 - \left( 1 + \frac{y}{1+x} \right)^{-\theta} \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + (1+x)^{-\theta} \int_1^{\infty} \left( 1 - \left( 1 + \frac{y}{1+x} \right)^{-\theta} \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

By restricting the function  $1 - (1+t)^{-\theta}$  to intervals  $(-1, 1)$  and  $[1, \infty)$ , and using its Taylor expansion, that is,

$$1 - (1+t)^{-\theta} = - \sum_{i=1}^{\infty} \binom{-\theta}{i} t^i,$$

for  $t \in (-1, 1)$ , where

$$\binom{-\theta}{i} = \frac{-\theta(-\theta-1)\cdots(-\theta-i+1)}{i!},$$

we get

$$\begin{aligned} C(x) &= (1+x)^{-\theta} \int_{1+x}^{\infty} \left( 1 - \left( \frac{1-x+y}{1+x} \right)^{-\theta} \right) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &\quad + (1+x)^{-\theta} \int_{x-1}^{1+x} \left( 1 - \left( \frac{1+\varphi(x-y)}{1+x} \right)^{-\theta} \right) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &\quad - (1+x)^{-\theta} \sum_{i=1}^{\infty} \binom{-\theta}{i} \frac{(-1)^i c(x)}{(1+x)^i} \int_1^{x-1} y^{i-\alpha(x)-1} dy \\ &\quad - (1+x)^{-\theta} \sum_{i=1}^{\infty} \binom{-\theta}{i} \frac{c(x)}{(1+x)^i} \int_1^{1+x} y^{i-\alpha(x)-1} dy \\ &\quad + (1+x)^{-\theta} \int_{1+x}^{\infty} \left( 1 - \left( 1 + \frac{y}{1+x} \right)^{-\theta} \right) \frac{c(x)}{y^{\alpha(x)+1}} dy. \end{aligned}$$

Let us put

$$\begin{aligned}
C_1(x) &:= \frac{\alpha(x)}{\theta} (1+x)^{\alpha(x)} \left[ \int_{1+x}^{\infty} \left( 1 - \left( \frac{1-x+y}{1+x} \right)^{-\theta} \right) \frac{dy}{y^{\alpha(x)+1}} \right. \\
&\quad \left. + \int_{1+x}^{\infty} \left( 1 - \left( 1 + \frac{y}{1+x} \right)^{-\theta} \right) \frac{dy}{y^{\alpha(x)+1}} \right] \\
C_2(x) &:= \frac{\alpha(x)}{\theta} (1+x)^{\alpha(x)} \int_{x-1}^{1+x} \left( 1 - \left( \frac{1+\varphi(x-y)}{1+x} \right)^{-\theta} \right) \frac{dy}{y^{\alpha(x)+1}} \\
C_3(x) &:= \alpha(x) (1+x)^{\alpha(x)-1} \int_{x-1}^{1+x} y^{-\alpha(x)} dy \\
C_4(x) &:= -\frac{\alpha(x)}{\theta} (1+x)^{\alpha(x)} \left[ \sum_{i=2}^{\infty} \binom{-\theta}{i} \frac{(-1)^i}{(1+x)^i} \int_1^{x-1} y^{i-\alpha(x)-1} dy \right. \\
&\quad \left. + \sum_{i=2}^{\infty} \binom{-\theta}{i} \frac{1}{(1+x)^i} \int_1^{1+x} y^{i-\alpha(x)-1} dy \right].
\end{aligned}$$

Hence, we find

$$\frac{\alpha(x)}{\theta c(x)} (1+x)^{\alpha(x)+\theta} C(x) = C_1(x) + C_2(x) + C_3(x) + C_4(x). \quad (3.25)$$

Further, by (3.6), we have

$$\begin{aligned}
C_1(x) &= \frac{2}{\theta} - \frac{\alpha(x) {}_2F_1(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, -1) + \alpha(x) {}_2F_1(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, 1)}{\theta(\alpha(x) + \theta)} \\
&\quad - \frac{\alpha(x) {}_2F_1\left(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, \frac{x-1}{1+x}\right) - \alpha(x) {}_2F_1(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, 1)}{\theta(\alpha(x) + \theta)}.
\end{aligned}$$

Next, by elementary computation, we have

$$C_3(x) = \frac{\alpha(x)}{1-\alpha(x)} \left( 1 - \left( \frac{1+x}{x-1} \right)^{\alpha(x)-1} \right)$$

in the case when  $\alpha(x) \neq 1$ ,

$$C_3(x) = \ln \left( \frac{1+x}{x-1} \right),$$

in the case when  $\alpha(x) = 1$ , and

$$\begin{aligned}
C_4(x) &= -\frac{\alpha(x)}{\theta} \left[ \sum_{i=1}^{\infty} \binom{-\theta}{2i} \frac{2}{2i - \alpha(x)} - \sum_{i=1}^{\infty} \binom{-\theta}{2i} \frac{2}{(2i - \alpha(x))(1+x)^{2i-\alpha(x)}} \right. \\
&\quad \left. + \sum_{i=2}^{\infty} \binom{-\theta}{i} \frac{(-1)^i}{i - \alpha(x)} \left( \left( \frac{x-1}{1+x} \right)^{i-\alpha(x)} - 1 \right) \right].
\end{aligned}$$

**Step 3.** In the third step we prove

$$\liminf_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} (1+x)^{\alpha(x)+\theta} \tilde{\mathcal{A}}V(x) = \liminf_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} (1+x)^{\alpha(x)+\theta} (A(x) + B(x) + C(x)) > 0.$$

First, by the mean value theorem, we have

$$\lim_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} (1+x)^{\alpha(x)+\theta} B(x) = 0, \quad (3.26)$$

and, by Lemma 3.2, we have

$$\lim_{x \rightarrow \infty} C_2(x) = 0 \quad (3.27)$$

and

$$\lim_{x \rightarrow \infty} C_3(x) = 0. \quad (3.28)$$

Further, since  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\} \leq \sup\{\alpha(x) : x \in \mathbb{R}\} < 2$ , from (3.5) and the dominated convergence theorem, it follows

$$\lim_{x \rightarrow \infty} \frac{\alpha(x) {}_2F_1\left(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, \frac{x-1}{1+x}\right) - \alpha(x) {}_2F_1(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, 1)}{\theta(\alpha(x) + \theta)} = 0 \quad (3.29)$$

and

$$\lim_{x \rightarrow \infty} \left[ - \sum_{i=1}^{\infty} \binom{-\theta}{2i} \frac{2}{(2i - \alpha(x))(1+x)^{2i-\alpha(x)}} + \sum_{i=2}^{\infty} \binom{-\theta}{i} \frac{(-1)^i}{i - \alpha(x)} \left( \left( \frac{x-1}{1+x} \right)^{i-\alpha(x)} - 1 \right) \right] = 0. \quad (3.30)$$

Thus, by combining (3.24) - (3.30), we have

$$\begin{aligned} & \liminf_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} (1+x)^{\alpha(x)+\theta} \tilde{\mathcal{A}}V(x) \\ &= \liminf_{x \rightarrow \infty} \left[ \frac{\alpha(x)}{c(x)} (1+x)^{\alpha(x)-1} \beta(x) - \frac{\alpha(x)}{\theta} \sum_{i=1}^{\infty} \binom{-\theta}{2i} \frac{2}{2i - \alpha(x)} + \frac{2}{\theta} \right. \\ & \quad \left. - \frac{\alpha(x) {}_2F_1(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, -1) + \alpha(x) {}_2F_1(\theta, \alpha(x) + \theta, 1 + \alpha(x) + \theta, 1)}{\theta(\alpha(x) + \theta)} \right]. \end{aligned}$$

Next, it can be proved that the function

$$\theta \mapsto -\frac{\alpha}{\theta} \sum_{i=1}^{\infty} \binom{-\theta}{2i} \frac{2}{2i - \alpha} + \frac{2}{\theta} - \frac{\alpha {}_2F_1(\theta, \alpha + \theta, 1 + \alpha + \theta, -1) + \alpha {}_2F_1(\theta, \alpha + \theta, 1 + \alpha + \theta, 1)}{\theta(\alpha + \theta)}$$

is strictly decreasing, hence we choose  $\theta$  close to zero. From, (3.2), (3.3) and (3.4), we get

$$\begin{aligned} & \lim_{\theta \rightarrow 0} \left( -\frac{\alpha}{\theta} \sum_{i=1}^{\infty} \binom{-\theta}{2i} \frac{2}{2i - \alpha} + \frac{2}{\theta} - \frac{\alpha {}_2F_1(\theta, \alpha + \theta, 1 + \alpha + \theta, -1) + \alpha {}_2F_1(\theta, \alpha + \theta, 1 + \alpha + \theta, 1)}{\theta(\alpha + \theta)} \right) \\ &= \pi \operatorname{ctg} \left( \frac{\pi \alpha}{2} \right). \end{aligned}$$

Now, the claim follows from condition (1.3). The case when  $x < 0$  is treated in the same way. Therefore, we have proved the desired result.  $\square$

In the case when  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$ , we also give an alternative, more probabilistic, proof of Theorem 1.1 (ii).

**Proof of Theorem 1.1 (ii).** Let  $\{X_n\}_{n \geq 0}$  be a Markov chain on the real line given by the transition kernel  $p(x, dy) := f_x(y - x)dy$ , where  $f_x(y)$  is the density function of the stable distribution with characteristic exponent  $p(x; \xi) = -i\beta(x)\xi + \gamma(x)|\xi|^{\alpha(x)}$ . Hence, the chain  $\{X_n\}_{n \geq 0}$  jumps from the state  $x$  by the stable distribution with the density function  $f_x(y)$ . By [San12a, Proposition 5.5 and Theorem 1.4], the chain  $\{X_n\}_{n \geq 0}$  is transient. Further, let  $\{X_n^m\}_{n \geq 0}$ ,  $m \in \mathbb{N}$ , be a sequence of Markov chains on the real line given by transition kernels  $p_m(x, dy) := f_x^m(y - x)dy$ ,  $m \in \mathbb{N}$ , where  $f_x^m(y)$  is the density function of the stable distribution with characteristic exponent  $p_m(x; \xi) := \frac{1}{m}p(x; \xi)$ . Then, by [San12b, Proposition 2.13], all chains  $\{X_n^m\}_{n \geq 0}$ ,  $m \in \mathbb{N}$ , are transient as well. Further, by [BS09], we have

$$X_{[m \cdot]}^m \xrightarrow{d} X^\alpha, \text{ as } m \rightarrow \infty,$$

where  $[x]$  denotes the integer part of  $x$  and  $\xrightarrow{d}$  denotes the convergence in the space of càdlàg functions equipped with the Skorohod topology. Hence, the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  can be approximated by càdlàg “versions” of chains  $\{X_n^m\}_{n \geq 0}$ ,  $m \in \mathbb{N}$ . Therefore, all we have to show is that this approximation keeps the transience property. But this fact follows from [San12b, Proposition 2.4].  $\square$

**Proof of Theorem 1.2.** We use a similar strategy as in Theorem 1.1 (ii). The proof is divided in three steps.

**Step 1.** In the first step we explain our strategy of the proof. Let  $\varphi \in C^2(\mathbb{R})$  be an arbitrary nonnegative function such that  $\varphi(x) = |x|$ , for  $|x| > 1$ , and  $\varphi(x) \leq |x|$ , for  $|x| \leq 1$ , and let  $\theta \in (1, \alpha)$  be arbitrary (recall that  $1 < \alpha = \inf\{\alpha(x) : x \in \mathbb{R}\}$ ) and let us define the function  $V : \mathbb{R} \rightarrow \mathbb{R}_+$  by the formula

$$V(x) := (\varphi(x))^\theta.$$

Clearly,  $V \in C^2(\mathbb{R})$  and the level set  $C_V(r) = \{x : V(x) \leq r\}$  is a compact set for all levels  $r \geq 0$ . Furthermore, since  $\theta < \inf\{\alpha(x) : x \in \mathbb{R}\}$ , we have

$$\left| \int_{\{|y| > 1\}} (V(x+y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \right| < \infty$$

for all  $x \in \mathbb{R}$ . Hence, by the relation (2.1),  $V \in \mathcal{D}(\tilde{\mathcal{A}})$  and for the function  $\tilde{\mathcal{A}}V(x)$  we can take the function  $\mathcal{A}^\alpha V(x)$ , where  $\mathcal{A}^\alpha$  is the infinitesimal generator of the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  given by (1.1).

In the sequel we show that there exists  $r_0 > 0$ , large enough, such that  $\tilde{\mathcal{A}}V(x) \leq -1$  for all  $x \in (C_V(r_0))^c$ . Clearly,  $\sup_{x \in C_V(r_0)} |\tilde{\mathcal{A}}V(x)| < \infty$ . Thus, the desired result follows from Theorem 2.3 (iii). In order to see this, since  $C_V(r) \uparrow \mathbb{R}$ , when  $r \nearrow \infty$ , it suffices to show that

$$\limsup_{\theta \rightarrow \alpha} \limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta} (\tilde{\mathcal{A}}V(x) + 1) < 0. \quad (3.31)$$

We have

$$\begin{aligned} \tilde{\mathcal{A}}V(x) + 1 &= \mathcal{A}V(x) + 1 = \beta(x)V'(x) + \int_{\{|y| \leq 1\}} (V(x+y) - V(x) - V'(x)y) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + \int_{\{|y| > 1\}} (V(x+y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy + 1. \end{aligned}$$

Let us define

$$\begin{aligned} A(x) &:= \beta(x)V'(x) + 1 \\ B(x) &:= \int_{\{|y| \leq 1\}} (V(x+y) - V(x) - V'(x)y) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ C(x) &:= \int_{\{|y| > 1\}} (V(x+y) - V(x)) \frac{c(x)}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

Hence, in order to prove (3.31) it suffices to prove

$$\limsup_{\theta \rightarrow \alpha} \limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta} (A(x) + B(x) + C(x)) < 0. \quad (3.32)$$

Furthermore, for  $x > 0$  large enough we have

$$\begin{aligned} A(x) &= \theta \beta(x) x^{\theta-1} + 1 \\ B(x) &= \int_{-1}^1 \left( (x+y)^\theta - x^\theta - \theta x^{\theta-1} y \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \end{aligned}$$

and

$$\begin{aligned} C(x) &= \int_{\{|y| > 1\}} \left( (\varphi(x+y))^\theta - x^\theta \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &= x^\theta \int_{-\infty}^{-x-1} \left( \left( -\frac{y}{x} - 1 \right)^\theta - 1 \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + x^\theta \int_{-x-1}^{-x+1} \left( \left( \frac{\varphi(x+y)}{x} \right)^\theta - 1 \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + x^\theta \int_{-x+1}^{-1} \left( \left( 1 + \frac{y}{x} \right)^\theta - 1 \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy \\ &\quad + x^\theta \int_1^\infty \left( \left( 1 + \frac{y}{x} \right)^\theta - 1 \right) \frac{c(x)}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

By restricting the function  $(1+t)^\theta - 1$  to intervals  $(-1, 1)$  and  $[1, \infty)$ , and using its Taylor expansion, that is,

$$(1+t)^\theta - 1 = \sum_{i=1}^{\infty} \binom{\theta}{i} t^i,$$

for  $t \in (-1, 1)$ , we get

$$\begin{aligned} C(x) &= x^\theta \int_{1+x}^\infty \left( \left( \frac{y}{x} - 1 \right)^\theta - 1 \right) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &\quad + x^\theta \int_{x-1}^{1+x} \left( \left( \frac{\varphi(x-y)}{x} \right)^\theta - 1 \right) \frac{c(x)}{y^{\alpha(x)+1}} dy \\ &\quad + x^\theta \sum_{i=1}^{\infty} \binom{\theta}{i} \frac{(-1)^i c(x)}{x^i} \int_1^{x-1} y^{i-\alpha(x)-1} dy \\ &\quad + x^\theta \sum_{i=1}^{\infty} \binom{\theta}{i} \frac{c(x)}{x^i} \int_1^x y^{i-\alpha(x)-1} dy \\ &\quad + x^\theta \int_x^\infty \left( \left( 1 + \frac{y}{x} \right)^\theta - 1 \right) \frac{c(x)}{y^{\alpha(x)+1}} dy. \end{aligned}$$

Let us put

$$\begin{aligned}
C_1(x) &:= \frac{\alpha(x)}{\theta} x^{\alpha(x)} \left[ \int_{1+x}^{\infty} \left( \left( \frac{y}{x} - 1 \right)^{\theta} - 1 \right) \frac{dy}{y^{\alpha(x)+1}} + \int_x^{\infty} \left( \left( 1 + \frac{y}{x} \right)^{\theta} - 1 \right) \frac{dy}{y^{\alpha(x)+1}} \right] \\
C_2(x) &:= \frac{\alpha(x)}{\theta} x^{\alpha(x)} \int_{x-1}^{1+x} \left( \left( \frac{\varphi(x+y)}{x} \right)^{\theta} - 1 \right) \frac{dy}{y^{\alpha(x)+1}} \\
C_3(x) &:= \alpha(x) x^{\alpha(x)-1} \int_{x-1}^x y^{-\alpha(x)} dy \\
C_4(x) &:= \frac{\alpha(x)}{\theta} x^{\alpha(x)} \left[ \sum_{i=2}^{\infty} \binom{\theta}{i} \frac{(-1)^i}{x^i} \int_1^{x-1} y^{i-\alpha(x)-1} dy + \sum_{i=2}^{\infty} \binom{\theta}{i} \frac{1}{x^i} \int_1^x y^{i-\alpha(x)-1} dy \right].
\end{aligned}$$

Hence, we find

$$\frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} C(x) = C_1(x) + C_2(x) + C_3(x) + C_4(x). \quad (3.33)$$

Further, by (3.6), we have

$$\begin{aligned}
C_1(x) &= -\frac{2}{\theta} + \frac{\alpha(x) {}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, 1) + \alpha(x) {}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, -1)}{\theta(\alpha(x) - \theta)} \\
&\quad + \frac{1}{\theta} \left( 1 - \left( \frac{x}{1+x} \right)^{\alpha(x)} \right) - \frac{\alpha(x) {}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, 1)}{\theta(\alpha(x) - \theta)} \\
&\quad + \alpha(x) \left( \frac{x}{1+x} \right)^{\alpha(x)-\beta} \frac{{}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, \frac{x}{1+x})}{\theta(\alpha(x) - \theta)}.
\end{aligned}$$

Next, by elementary computation, we have

$$C_3(x) = \frac{\alpha(x)}{1 - \alpha(x)} \left( 1 - \left( \frac{x}{x-1} \right)^{\alpha(x)-1} \right)$$

and

$$\begin{aligned}
C_4(x) &= \frac{\alpha(x)}{\theta} \left[ \sum_{i=1}^{\infty} \binom{\theta}{2i} \frac{2}{2i - \alpha(x)} - \sum_{i=1}^{\infty} \binom{\theta}{2i} \frac{2}{(2i - \alpha(x)) x^{2i-\alpha(x)}} \right. \\
&\quad \left. + \sum_{i=2}^{\infty} \binom{\theta}{i} \frac{(-1)^i}{i - \alpha(x)} \left( \left( \frac{x-1}{x} \right)^{i-\alpha(x)} - 1 \right) \right].
\end{aligned}$$

**Step 3.** In the third step we prove

$$\limsup_{\theta \rightarrow \alpha} \limsup_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} \tilde{\mathcal{A}}V(x) = \limsup_{\theta \rightarrow \alpha} \limsup_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} (A(x) + B(x) + C(x)) < 0.$$

First, by the mean value theorem, we have

$$\lim_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} B(x) = 0, \quad (3.34)$$

and, by Lemma 3.2, we have

$$\lim_{x \rightarrow \infty} C_2(x) = 0 \quad (3.35)$$

and

$$\lim_{x \rightarrow \infty} C_3(x) = 0. \quad (3.36)$$

Further, since  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\} \leq \sup\{\alpha(x) : x \in \mathbb{R}\} < 2$ , from (3.5) and the dominated convergence theorem, it follows

$$\begin{aligned} \lim_{x \rightarrow \infty} \left[ \frac{1}{\theta} \left( 1 - \left( \frac{x}{1+x} \right)^{\alpha(x)} \right) - \frac{\alpha(x) {}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, 1)}{\theta(\alpha(x) - \theta)} \right. \\ \left. + \alpha(x) \left( \frac{x}{1+x} \right)^{\alpha(x)-\beta} \frac{{}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, \frac{x}{1+x})}{\theta(\alpha(x) - \theta)} \right] = 0 \end{aligned} \quad (3.37)$$

and

$$\lim_{x \rightarrow \infty} \left[ - \sum_{i=1}^{\infty} \binom{\theta}{2i} \frac{2}{(2i - \alpha(x))x^{2i-\alpha(x)}} + \sum_{i=2}^{\infty} \binom{\theta}{i} \frac{(-1)^i}{i - \alpha(x)} \left( \left( \frac{x-1}{x} \right)^{i-\alpha(x)} - 1 \right) \right] = 0. \quad (3.38)$$

Thus, by combining (3.32) - (3.38), we have

$$\begin{aligned} & \limsup_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} \tilde{\mathcal{A}}V(x) \\ &= \limsup_{x \rightarrow \infty} \left[ \frac{\alpha(x)}{c(x)} x^{\alpha(x)-1} \beta(x) + \frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} + \frac{\alpha(x)}{\theta} \sum_{i=1}^{\infty} \binom{\theta}{2i} \frac{2}{2i - \alpha(x)} - \frac{2}{\theta} \right. \\ & \quad \left. + \frac{\alpha(x) {}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, -1) + \alpha(x) {}_2F_1(-\theta, \alpha(x) - \theta, 1 + \alpha(x) - \theta, 1)}{\theta(\alpha(x) - \theta)} \right] \\ &= \limsup_{x \rightarrow \infty} \left( \frac{\alpha(x)}{c(x)} x^{\alpha(x)-1} \beta(x) + \frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} + E(\alpha(x), \theta) \right). \end{aligned}$$

Clearly, because of the term  $x^{\alpha(x)-\theta}$ , we choose  $\theta$  close to  $\alpha$ . Now, from (1.3), it follows

$$\limsup_{\theta \rightarrow \alpha} \limsup_{x \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} x^{\alpha(x)-\theta} \tilde{\mathcal{A}}V(x) < 0.$$

The case when  $x < 0$  is treated in the same way. Therefore, we have proved the desired result.  $\square$

**Proof of Corollary 1.3.** In the case when  $\alpha \neq 2$ , the claim easily follows from Theorem 1.1. Further, in the case when  $\alpha = 2$ , that is, in the Brownian motion case, the corresponding infinitesimal generator is given by  $\mathcal{A}^2 f(x) = \gamma f''(x)$  (recall that the symbol (characteristic exponent) is given by  $p(\xi) = \gamma |\xi|^2$ ) and clearly  $C^2(\mathbb{R}) \subseteq \mathcal{D}(\tilde{\mathcal{A}})$ . Thus, for any  $f \in C^2(\mathbb{R})$ , for the function  $\tilde{\mathcal{A}}f(x)$  we can take the function  $\mathcal{A}^2 f(x)$ . Now, by taking again  $V(x) = \log(1 + \varphi(x))$  for the test function, where  $\varphi \in C^2(\mathbb{R})$  is an arbitrary nonnegative function such that  $\varphi(x) = |x|$  for all  $|x| > 1$ , we get

$$\tilde{\mathcal{A}}V(x) = \mathcal{A}^2 V(x) = \frac{-\gamma}{(1 + |x|)^2}$$

for all  $|x| > 1$ , that is, the Brownian motion is recurrent.  $\square$



**Proof of Theorem 2.4.** Let  $\theta \in (1, \alpha)$  be arbitrary (recall that  $1 < \alpha = \inf_{x \in \mathbb{R}} \alpha(x)$ ). Further, let there exist a Borel measurable function  $f(x) \geq 1$  and strictly increasing, nonnegative and convex function  $\phi(x)$ , such that  $|x|^\theta = \phi(f(x))$  for all  $|x|$  large enough, then, by [MT93b, Theorem 5.3 (ii)], the stable-like process  $\{X_t^\alpha\}_{t \geq 0}$  is  $f$ -ergodic if

$$\tilde{\mathcal{A}}|x|^\theta \leq -f(x)$$

holds for all  $|x|$  large enough. Now, by repeating the proof of Theorem 1.2 and applying condition (2.2), we have

$$\begin{aligned} & \limsup_{|x| \rightarrow \infty} \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta} (\tilde{\mathcal{A}}V(x) + f(x)) \\ &= \limsup_{|x| \rightarrow \infty} \left( \operatorname{sgn}(x) \frac{\alpha(x)}{c(x)} |x|^{\alpha(x)-1} \beta(x) + \frac{\alpha(x)}{\theta c(x)} |x|^{\alpha(x)-\theta} f(x) + E(\alpha(x), \theta) \right) < 0. \end{aligned}$$

Therefore, we have proved the desired result.  $\square$

## Acknowledgement

The author would like to thank the anonymous reviewer for careful reading of the paper and for helpful comments that led to improvement of the presentation.

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